Professor F. Grillot

Midterm Exam

EE 270 - Applied Quantum Mechanics

Wednesday Nov. 22, 2017, 8.00 AM-9.50 AM

General instructions

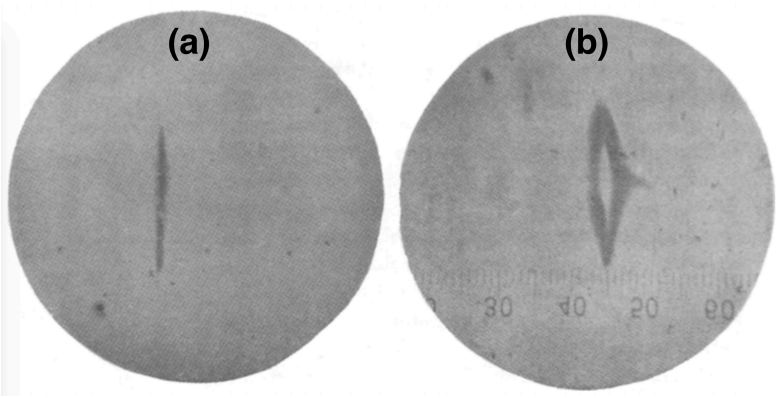
This is an open book exam.

When grading, we are focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Exercise I (5 points)
Explain in a few sentences the third postulate of quantum mechanics. What is the wavefunction collapse ?

Exercise II (10 points)
(a) Describe in a few sentences the Stern-Gerlach experiment.
(b) The result of the experiment originally published in [Zeitschrift für Physik, Vol. 9, pp 349-352, 1922] is depicted below. Explain what is unveiled and the difference between (a) and (b).



Exercise III (5 points)
A three-dimensional potential for a particle of mass m is of the form

$$V(x, y, z) = m\omega^2(x^2 + y^2 + 2z^2)$$

Write the potential as a matrix. Give the energy levels of the particle.

Exercise IV (10 points)
(a) What are the eigenvalues of the momentum operator $\hat{p}_x$?
(b) Determine the corresponding eigenfunctions ? Conclusion.

Exercise V (15 points)
The Hamiltonian of a particle mass m moving in a one-dimensional harmonic oscillator potential can be written

$$\hat{H} = \hbar\omega\left(\hat{a^\dagger}\hat{a} + \frac{1}{2}\right)$$

where ω is the angular frequency of oscillation and the operator

$$\hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left[\hat{x} + i\frac{\hat{p}}{m\omega}\right]$$

satisfies the commutation relations $[\hat{a}, \hat{a^\dagger}] = 1$, and $[\hat{a}, \hat{a}] = [\hat{a^\dagger}, \hat{a^\dagger}] = 0$.
(a) Show that the $n = 0$ ground-state $|0\rangle$ is defined by $\hat{a}\,|0\rangle = 0$.
(b) Find the normalized ground-state wave function $\psi_0(x)$. In answering this question, you may wish to use the standard integral $\int_{-\infty}^{+\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$.
(c) What is the energy E_0 of the ground state level ? Give a physical interpretation of this result (Hint : use Heisenberg uncertainty relation). Compare your analysis with the classical harmonic oscillator.

Exercise VI (20 points)

We consider a quantum system on which we measure a physical quantity A, associated with the observable $\hat{A}$. The set $\{|\psi_n\rangle\}$, with $n = 0, 1, ...$, is a orthonormal eigenbasis of $\hat{A}$ and the corresponding eigenvalues are denoted a_n. These eigenvalues are ordered by increasing values ($a_0 \leq a_1 \leq a_2 \leq ...$) and we are looking for an upper bound of the smallest eigenvalue a_0.

Let $|\psi\rangle$ be a vector of the Hilbert space $\mathcal{E}$. We expand $|\psi\rangle$ on the basis $|\psi_n\rangle$:

$$|\psi\rangle = \sum_n C_n |\psi_n\rangle$$

(a) Give the expression of the coefficients C_n in terms of $|\psi\rangle$ and $|\psi_n\rangle$.

(b) The norm of $|\psi\rangle$ is equal to 1. Write the corresponding relation satisfied by the coefficients C_n.

(c) Write the average value $\langle\psi|\hat{A}|\psi\rangle$ as a sum involving the eigenvalues a_n and the coefficients C_n.

(d) Using $a_0 \leq a_n$ for all n, prove the relation : $a_0 \leq \langle\psi|\hat{A}|\psi\rangle$.

Problem (35 points)

We consider an ammonia molecule and we restrict ourselves to the subspace $\mathcal{E}_0$ formed by the linear combinations of the lowest energy states $|\psi_S\rangle$ and $|\psi_A\rangle$. In the basis $\{|\psi_S\rangle, |\psi_A\rangle\}$ the Hamiltonian of the molecule is such as :

$$\hat{H} = \begin{pmatrix} E_0 - A & 0 \\ 0 & E_0 + A \end{pmatrix}$$

We define the operator $\hat{X}$ associated with the "disposition with respect to the center"

$$\hat{X} = d \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where d is a fixed, known parameter. A physical or chemical process (that we will not describe) produces the molecules always in the state

$$|\psi(0)\rangle = \cos\theta\, |\psi_S\rangle + \sin\theta e^{i\phi} |\psi_A\rangle$$

where θ is between 0 and $\pi/2$, and the phase ϕ between 0 and 2π. We are now looking for an experimental way of determining these two parameters with a very good precision. In this purpose we take 3N ammonia molecules all prepared in the state $|\psi(0)\rangle$ (with $N >> 1$).

1. For N molecules among the $3N$ available, we perform an energy measurement at time $t = 0$.
(a) Calculate $\langle E \rangle$ for the state $|\psi(0)\rangle$.
(b) What are the possible results of an individual energy measurement ?
(c) Give the probability for each result and re-express $\langle E \rangle$. Conclusion.
(d) Can we determine θ and ϕ without ambiguity ?

2. For N molecules among the $2N$ remaining, we perform a measurement of X at time $t = 0$.
(a) What are the possible results in a measurement of X ?
(b) Calculate the average value $\langle X \rangle_0$ for the state $|\psi(0)\rangle$.
(c) Do we get further information on θ and ϕ ? Is it now possible to determine unambiguously these two parameters ?

Bonus (10 points)
3. We let the N remaining molecules undergo a free evolution for a duration T, and we then perform a measurement of X on each of these molecules.
(a) Write the state $|\psi(T)\rangle$ of these molecules just before the measurement of X.
(b) Let T be such that $AT/\hbar = \pi/4$. Calculate the average value $\langle X \rangle_T$ of the results for the state $|\psi(T)\rangle$.
(c) Show that the initially unknown state $|\psi(0)\rangle$ is now fully determined if one combines the results of the three sets of measurements.